U.S. DEPARTMENT OF COMMERCE
National Technical Information Service
PB-287 573

Attrition in Distributed Delay Models

Corvallis Environmental Research Lab, OR

Sep 77

TECHNICAL REPORT DATA (Please read Instructions on the reverse before completing)		
1. REPORT NO. EPA-600/J-77-135	2. JOURNAL ARTICLE	3. PB 287573
4. TITLE AND SUBTITLE Attrition in Distributed Delay Models		5. REPORT DATE 6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) John VanSickle		8. PERFORMING ORGANIZATION REPORT NO.
9. PERFORMING ORGANIZATION NAME AND ADDRESS Environmental Research Laboratory - Corvallis ORD EPA Corvallis, Oregon 97330		10. PROGRAM ELEMENT NO. 1BA608 11. CONTRACT/GRANT NO.
12. SPONSORING AGENCY NAME AND ADDRESS Same as above		13. TYPE OF REPORT AND PERIOD COVERED Journal Article 14. SPONSORING AGENCY CODE EPA/600/02

15. SUPPLEMENTARY NOTES
IEEE Transactions on Systems, Man, and Cybernetics, Vol. SMC-7 No. 9, Sept 1977
Pages 635-638

16. ABSTRACT

Distributed delay models are extended to deal with loss of entities from within the process being modeled, analysis of a time-invariant delay shows that the statistics of individual transit times through a delay process can be very sensitive to attrition incurred during the delay.

17. KEY WORDS AND DOCUMENT ANALYSIS

a. DESCRIPTORS	b. IDENTIFIERS/OPEN ENDED TERMS	c. COSATI Field/Group
Attrition Casualties Comminution Delay Models		05/K

REPRODUCED BY
NATIONAL TECHNICAL
INFORMATION SERVICE
U.S. DEPARTMENT OF COMMERCE
SPRINGFIELD, VA. 22161

18. DISTRIBUTION STATEMENT	19. SECURITY CLASS (This Report)	21. NO. OF PAGES
Release to Public	UNCLASSIFIED	6
	20. SECURITY CLASS (This page)	22. PRICE
	UNCLASSIFIED	PC A02 MF A01

EPA Form 2220-1 (Rev. 4-77) PREVIOUS EDITION IS OBSOLETE

EPA-600/J-77-135
JOURNAL ARTICLE

Attrition in Distributed Delay Models

JOHN VANSICKLE

Abstract—**Distributed time delay models are extended to deal with loss of entities from within the process being modeled. Analysis of a time-invariant distributed delay shows that the statistics of individual transit times through a delay process can be very sensitive to attrition incurred during the delay.**

I. Introduction

DISTRIBUTED time delay models have proven to be useful tools in simulating the aggregate flow of entities through a process or developmental phase. Forrester [2] and Manetsch [7] among others, have employed distributed delays in industrial and economic models to describe delays in transport, production, land development, etc. The delay model has also been widely used by hydrologists in computing hydrographs for watershed runoff [9]. In population models, distributed delays can be used to "grow" organisms through developmental stages in their life cycles [1], [2], [7]. The theory and practice of distributed delay modeling have been comprehensively presented in [2], [6], and [7].

One important feature of distributed delays is their conservation of flow—all entities entering the delay process will eventually emerge. The realism and utility of delay models can be greatly enhanced by providing for a net gain or loss of flow during the delay process. In biological populations, for example, individuals of any developmental stage can be lost to mortality or emigration, or immigrants may add to the numbers in the stage. Transportation models should be able to represent losses of goods due to damage or theft during a transport delay.

This paper extends distributed delay models to deal with a simple and widely applicable description of loss of flow from the delay process. The model assumes that the loss rate is directly proportional to the number of entities in the delay. The proportional loss, or *attrition*, model has already been applied to discrete time delays in linear flow population models [5].

Assumption of an attrition model requires only minor modifications of existing theory and simulation techniques. However, I will show that the statistics of time of emergence from a distributed delay can be markedly changed by attrition incurred during the delay.

II. Distributed Delays

A brief summary of the theory of distributed delays will be helpful. The most important equations are reproduced, and the reader may consult [6], [7], or [8] for details of their development. The notation of [7] will be used throughout.

A time-invariant distributed delay can be modeled with a set of k cascaded delay stages, as illustrated in Fig. 1. The equations for storage in the delay process are given by

$$\begin{aligned}\frac{dQ_1(t)}{dt} &= x(t) - r_1(t)\\ \frac{dQ_2(t)}{dt} &= r_1(t) - r_2(t)\\ &\vdots\\ \frac{dQ_k(t)}{dt} &= r_{k-1}(t) - r_k(t)\end{aligned} \tag{1}$$

where $r_k(t) = y(t)$ is the flow rate out of the process and $x(t)$ is the input rate. The dynamics of the delay model are specified by assuming that

$$r_i(t) = \frac{k}{\mathrm{DEL}} Q_i(t), \qquad i = 1, 2, \cdots, k. \tag{2}$$

The parameter DEL in (2) can easily be shown to be the average length of the delay imposed by the process.

Using (2), the system of equations (1) can be written either strictly in terms of flow rates or in terms of storages. In either form, the equations comprise a time-invariant linear system whose analysis is straightforward.

In particular, if an impulse input ($x(t) = \delta(t)$) is applied, the response of the system is given by[1]

$$y_I(t) = \left[\frac{k}{\mathrm{DEL}}\right]^k \frac{t^{(k-1)}}{(k-1)!} \exp\left(\frac{-kt}{\mathrm{DEL}}\right). \tag{3}$$

Manetsch [6], [7] has shown that this input–output relationship has a valuable probabilistic interpretation as an aggregate representation of the behavior of many entities moving through the delay process.

A large number of entities which enter the delay process simultaneously, i.e., an impulse input, will emerge distributed in time, due to individual differences in processing rate. The function $y_I(t)$ can thus be viewed as a density function for the random variable τ, the transit time of an entity in the delay. Equation (3) defines the Erlang family of density functions, and the mean and variance of τ are given by

$$\mu_\tau = \mathrm{DEL} \tag{4}$$

$$\sigma_\tau^2 = \frac{\mathrm{DEL}^2}{k}. \tag{5}$$

Equations (4) and (5) can be used to parameterize a delay model from transit time statistics of the process being modeled.

The interpretation of $y_I(t)$ as a probability density suggests that equivalent models can be formulated in the

Manuscript received October 20, 1976; revised April 4, 1977. This work was supported in part by the National Research Council.

The author is with the Ecosystems Analysis and Modeling Branch, Environmental Protection Agency, Corvallis Environmental Research Laboratory, Corvallis, OR 97330.

[1] The factor $[k/\mathrm{DEL}]^k$ is mistakenly inverted in (5) of [7].

Copyright © 1977 by The Institute of Electrical and Electronics Engineers, Inc.
Printed in U.S.A. Annals No. 709SM001

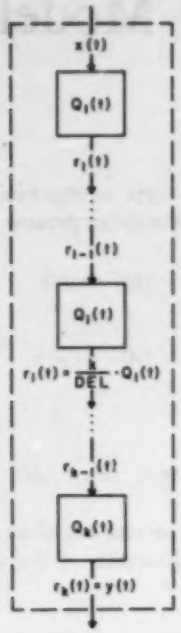

Fig. 1. k-stage time-invariant distributed delay (adapted from [8]). $r_i(t)$ = flow rate out of stage i. $Q_i(t)$ = storage in stage i.

language of stochastic processes. Such models have been devised for biological populations. Takahashi [13] and Read and Ashford [12] divide the life histories of organisms into a series of stages. Development out of a stage is modeled as a Poisson process, which leads to the same single-stage dynamics as (1). Both papers also model mortality loss from each stage as a Poisson process. In their model. Gordon *et al.* [3] use deterministic equations identical to (1) for passage through growth stages, but they too employ a Poisson process for mortality.

These mortality models are the stochastic equivalents of the attrition model discussed in the next section, and the resulting dynamics are the same as those seen in the delay system. In particular, Takahashi [13] uses the stochastic framework to derive formulas identical to (11), (12), and (15), which are the central results of this paper.

III. Modeling Attrition

In most applications, the stages in a distributed delay do not correspond to recognizable stages in the modeled process, since k is determined empirically from (5). For this reason, it is best to model gains or losses from the system at the level of the full delay process.

Let

$$Q(t) = \sum_{i=1}^{k} Q_i(t)$$

be the total storage in the delay. Then summing (1) yields

$$\frac{dQ(t)}{dt} = x(t) - y(t). \tag{6}$$

Terms for net gain or loss of individuals can now be added to (6). We will consider a simple and useful model for loss from the process. This model states that the loss rate from the delay is proportional to the total storage, and it will be referred to as an *attrition* model. Equation (6) becomes

$$\frac{dQ(t)}{dt} = x(t) - y(t) - \text{AR}\ (t) \cdot Q(t), \tag{7}$$

where AR (t) is the instantaneous attrition rate. The equation is the sum of a modified form of (1):

$$\begin{aligned}
\frac{dQ_1}{dt} &= x(t) - r_1(t) - \text{AR}\ (t) \cdot Q_1(t) \\
\frac{dQ_2}{dt} &= r_1(t) - r_2(t) - \text{AR}\ (t) \cdot Q_2(t) \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
\frac{dQ_k}{dt} &= r_{k-1}(t) - y(t) - \text{AR}\ (t) \cdot Q_k(t).
\end{aligned} \tag{8}$$

Equations (8) describe the dynamics of a k-stage distributed delay with attrition. In applications for which the k delay stages have real-world counterparts, one could easily make AR (t) a stage-specific loss rate.

For simulation purposes it is convenient to write (8) solely in terms of flow rates, using (2). The equation for the ith stage is

$$\frac{dr_i(t)}{dt} = \frac{k}{\text{DEL}}\left[r_{i-1}(t) - \left(1 + \text{AR}\ (t)\frac{\text{DEL}}{k}\right) r_i(t)\right], \quad i = 1, 2, \cdots, k \tag{9}$$

with $r_0(t) = x(t)$ and $r_k(t) = y(t)$.

Simulation of (9) is easily accomplished by modifying the delay subroutines of existing languages such as Fordyn [4] or Dynamo [11]. If the mean delay, DEL, is time varying, then the derivation of Manetsch [7] can be altered to include attrition. In this case, substitution of (2), with DEL a function of time, into (8) results in the following expression for the flow rates:

$$\frac{dr_i(t)}{dt} = \frac{k}{\text{DEL}\ (t)}\left[r_{i-1}(t) - \left(1 + \frac{\text{AR}\ (t) \cdot \text{DEL}\ (t)}{k} + \frac{d\ \text{DEL}\ (t)}{dt}\right) r_i(t)\right], \quad i = 1, 2, \cdots, k. \tag{10}$$

The term in (10) for attrition can be added to Manetsch's [7] subroutine VDEL.

IV. Analysis of Delays with Attrition

It is interesting to examine the effects of attrition on the dynamics of a distributed delay. At first glance, it might seem that the pattern of emergence from a delay would be unaffected by attrition, since the loss rate of AR is assigned equally to all entities in the delay. However, this is not the case.

Suppose that both DEL and AR are constants. Then the system of equations (9) is time invariant. The Appendix derives the response of the system (9) to an impulse input and normalizes the response so that it can be interpreted as a probability density function. Let τ^* be the random variable for the transit times of survivors of the delay. The Appendix demonstrates that τ^* has a gamma distribution, and it compares the shape of this distribution with the distributions (3) seen when there is no attrition.

The moments of gamma densities are well known [10], and the mean and variance of τ^* are given by

$$\mu_{\tau^*} = \text{DEL}\,\frac{k}{(\text{DEL}\cdot\text{AR}+k)} \tag{11}$$

$$\sigma^2_{\tau^*} = \frac{\text{DEL}^2}{k}\left(\frac{k}{\text{DEL}\cdot\text{AR}+k}\right)^2. \tag{12}$$

Notice that (11) and (12) reduce to (4) and (5) for AR = 0.

Comparison of (4) with (11) and (5) with (12) shows that attrition in the delay process in fact *reduces* the mean and variance of transit times. This effect is illustrated in Fig. 2, which is a plot, for three different attrition losses, of the transit-time distributions of survivors of a three-stage delay. The figure shows that the mean delay for survivors of a 90 percent attrition loss is less than half the mean delay when there is no attrition. Attrition losses in excess of 90 percent are frequently encountered when one models the early developmental stages of many fish and insect species.

The reduction of the mean and variance in delay times is further aggravated by decreasing k. The role of k is clarified by noting that the coefficient of variation of τ^*, and of τ, is given by

$$\frac{\sigma_\tau}{\mu_\tau} = \frac{\sigma_{\tau^*}}{\mu_{\tau^*}} = \frac{1}{\sqrt{k}}. \tag{13}$$

Thus k is a measure of the dispersion in time of entities as they pass through the delay.

As a result, the mean and variance of τ^* are most sensitive to attrition when entities emerge from the delay greatly dispersed in time. At the other extreme, as k gets large, the entities emerge as a narrow pulse in time ($(\sigma/\mu) \simeq 0$), and $\mu_\tau \simeq \mu_{\tau^*}$.

These results lead to a simple explanation for the reduction in mean and variance. Attrition takes a greater toll of those entities which take longer to pass through the delay. Thus the distribution of transit times for survivors is skewed toward smaller values. It is clear that this principle applies not only to the system (9), but to any stochastic delay process with proportional loss along the delay.

V. Estimation of Parameters

In addition to (11) and (12), one further formula is needed to estimate k, DEL, and AR from an empirical distribution of transit times. Define ε as the fraction of the total input which survives to eventually emerge from the delay:

$$\varepsilon = \frac{\int_0^\infty y(t)\,dt}{\int_0^\infty x(t)\,dt}. \tag{14}$$

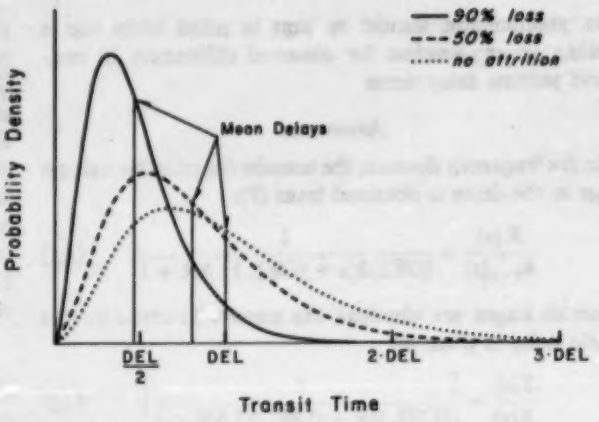

Fig. 2. Density functions, with mean values indicated, for transit times through three-stage distributed delay.

The Appendix shows that if $x(t) = \delta(t)$, then

$$\varepsilon = \left[\frac{k}{\text{DEL}\cdot\text{AR}+k}\right]^k. \tag{15}$$

Since an arbitrary input $x(t)$ can be written as a sum of impulses, and the delay model is linear, it follows that (15) is true for any input function.

The model parameters are estimated by observing transit times of entities through the process of interest and obtaining estimates for ε, μ_{τ^*}, and $\sigma^2_{\tau^*}$. The following equations can be derived from (11), (12), and (15):

$$k = \frac{\sigma^2_{\tau^*}}{\mu^2_{\tau^*}} \tag{16}$$

$$\text{DEL} = \mu_{\tau^*}\varepsilon^{-1/k} \tag{17}$$

$$\text{AR} = k\left[\frac{1}{\mu_{\tau^*}} - \frac{1}{\text{DEL}}\right]. \tag{18}$$

Equation (15) has another useful property. Manetsch [7] points out that, as $k \to \infty$, the distributed delay becomes a discrete delay of length DEL. It is not hard to show that, as $k \to \infty$, $\varepsilon \to \exp(-\text{AR}\cdot\text{DEL})$. This is clearly the fraction of the total input which should emerge from a discrete delay of length DEL and attrition rate AR [5]. When k is large, this relationship is an alternative to (18) in estimating AR. Equation (17) provides a quick estimate of the reduction of mean delay time due to attrition, since it can be rewritten as

$$\frac{\mu_{\tau^*}}{\mu_\tau} = \varepsilon^{1/k}. \tag{19}$$

VI. Summary and Conclusion

This paper has developed a useful extension of distributed delay models to deal with loss of flow from the delay. Analysis of the model with time-invariant parameters results in a graphic illustration of the effects of attrition. When individuals pass through a developmental stage or delay process at widely differing rates, the statistics of their aggregate behavior can be significantly altered by attrition.

This phenomenon should be kept in mind when one is seeking an explanation for observed differences in real-world process delay times.

Appendix

In the frequency domain, the transfer function for a single stage in the delay is obtained from (9):

$$\frac{R_i(s)}{R_{i-1}(s)} = \frac{1}{(\mathrm{DEL}/k)s + (\mathrm{DEL}/k)\,\mathrm{AR} + 1}. \quad \text{(A1)}$$

Since all stages are identical, the transfer function for the whole delay is given by

$$\frac{Y(s)}{X(s)} = \left[\frac{1}{(\mathrm{DEL}/k)s + (\mathrm{DEL}/k)\,\mathrm{AR} + 1}\right]^k. \quad \text{(A2)}$$

The right-hand side of (A2) is also the Laplace transform of the impulse response $y_t(t)$; so taking an inverse transform yields

$$y_t(t) = \left[\frac{k}{\mathrm{DEL}}\right]^k \frac{t^{(k-1)}}{(k-1)!} \exp\left[-\left(\mathrm{AR} + \frac{k}{\mathrm{DEL}}\right)t\right], \quad t \geq 0. \quad \text{(A3)}$$

Equation (A3) is normalized by defining ε so that

$$1 = \frac{1}{\varepsilon}\int_0^\infty y_t(t)\,dt. \quad \text{(A4)}$$

Since $\int_0^\infty x(t)\,dt = \int_0^\infty \delta(t)\,dt = 1$, the constant ε is simply the fraction of the "quantity" in the impulse input which survives attrition to eventually emerge from the delay.

The integral in (A4) is a gamma function whose evaluation is straightforward with k as an integer [10]:

$$\varepsilon = \int_0^\infty y_t(t)\,dt = \left[\frac{k}{\mathrm{DEL}\cdot\mathrm{AR} + k}\right]^k. \quad \text{(A5)}$$

Thus the function $(1/\varepsilon)y_t(t)$ can be viewed as a gamma density function for the transit time τ^* of an entity which survives the delay.

The shape of the transit time distributions with and without attrition may be compared using the expressions for $y_t(t)$ given in (3) and (A3). Equation (3) is the Erlang family of densities, which have been plotted for various k values in [2], [4], and [7]. It can be seen that (3) and (A3) are identical, except for the factor $\exp[-\mathrm{AR}\cdot t]$ in (A3). This results in Erlang-like densities for τ^*, with the right-handed tail seen in the Erlang functions progressively discounted by the factor $\exp[-\mathrm{AR}\cdot t]$. A transit-time density function for τ^* will appear more symmetrical about the mean than the corresponding function for τ. This effect is clearly seen in Fig. 2, where the density function with the highest attrition rate appears closest to a normal density.

Acknowledgment

I would like to thank Dr. T. J. Manetsch for his suggestion that distributed delays be extended to include attrition.

References

[1] M. H. Abkin and T. J. Manetsch, "A development planning oriented model of the agricultural economy of southern Nigeria," *IEEE Trans. Systems, Man, and Cybern.*, vol. SMC-2, pp. 472–486.
[2] J. W. Forrester, *Industrial Dynamics*. Cambridge, MA: MIT Press, 1961.
[3] G. Gordon, M. O'Callaghan, and G. M. Tallis, "A deterministic model for the life cycle of a class of internal parasites of sheep," *Math. Bio. Sci.*, vol. 8, pp. 209–226, 1970.
[4] R. Llewellyn, *Fordyn—An Industrial Dynamics Simulator*. Raleigh: North Carolina State University, 1965.
[5] E. R. Lewis, "Application of discrete and continuous network theory to linear population models," *Ecology*, vol. 57, pp. 33–47, 1976.
[6] T. J. Manetsch, "Transfer function representation of the aggregate behavior of a class of economic processes," *IEEE Trans. Automatic Control*, vol. AC-11, pp. 693–698, 1966.
[7] ——, "Time-varying distributed delays and their use in aggregate models of large systems," *IEEE Trans. Systems, Man, and Cybern.*, vol. SMC-6, pp. 547–553, 1976.
[8] T. J. Manetsch and G. L. Park, *System Analysis and Simulation with Applications to Economic and Social Systems*, Part II. East Lansing: Michigan State University (Division of Engineering Research), 1974.
[9] D. E. Overton and M. E. Meadows, *Stormwater Modeling*. New York: Academic Press, 1976.
[10] A. Papoulis, *Probability, Random Variables and Stochastic Processes*. New York: McGraw-Hill, 1965.
[11] A. L. Pugh, III, *Dynamo Users' Manual*. Cambridge, MA: MIT Press, 1963.
[12] K. L. Q. Read and J. R. Ashford, "A system of models for the life cycle of a biological organism," *Biometrika*, vol. 55, pp. 211–221, 1968.
[13] M. Takahashi, "Theoretical basis for cell cycle analysis," *J. Theor. Biol.*, vol. 18, pp. 195–209, 1968.